Bivariate transformation method

$$(Y_1, Y_2) \sim f_{Y_1, Y_2}(y_1, y_2)$$

$$U_1 = h_1(Y_1, Y_2) \qquad U_2 = h_2(Y_1, Y_2)$$

$$f_{U_1, U_2}(u_1, u_2) = f_{Y_1, Y_2}(h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2))\,|J|$$

$$J = \det \begin{array}{cc} \left[\begin{array}{cc} \frac{dy_1}{du_1} & \frac{dy_2}{du_1} \\ \frac{dy_1}{du_2} & \frac{dy_2}{du_2} \end{array}\right] & \begin{array}{c} u_1 \\ u_2 \end{array} \\ \begin{array}{cc} \quad y_1 & \qquad y_2 \end{array} & \end{array}$$

$$Y_1 \sim n(0,1) \qquad Y_2 \sim n(0,1)$$

$$f_{Y_1, Y_2}(y_1, y_2) = f_{Y_1}(y_1) \times f_{Y_2}(y_2)$$

$$= \underbrace{\left(\frac{e^{-\frac{y_1^2}{2}}}{\sqrt{2\pi}}\right)\left(\frac{e^{-\frac{y_2^2}{2}}}{\sqrt{2\pi}}\right)}$$

$$U_1 = Y_1 + Y_2 \Rightarrow Y_1 = U_1 - Y_2 \Rightarrow 2Y_1 = U_1 + U_2 \Rightarrow h_1^{-1}(u_1, u_2) = \frac{U_1 + U_2}{2} \qquad -\infty < u_1, u_2 < \infty$$

$$U_2 = Y_1 - Y_2 \Rightarrow Y_2 = Y_1 - U_2 \Rightarrow 2Y_2 = U_1 - U_2 \Rightarrow h_2^{-1}(u_1, u_2) = \frac{U_1 - U_2}{2}$$

$$f_{U_1, U_2}(u_1, u_2) = f_{Y_1, Y_2}(h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2))\,|J|$$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{2} \times \frac{1}{2} - \frac{1}{2} \times \frac{1}{2} = -\frac{1}{2}$$

$$=$$

Sub in h_1^{-1}, h_2^{-1}

$Y_1 \sim exp(\beta)$ $Y_2 \sim exp(\beta)$ independent

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{\beta^2} e^{-\frac{y_1+y_2}{\beta}}$$

$$U = \frac{Y_1}{Y_1+Y_2} \Rightarrow Y_1 = UW \Rightarrow h_1^{-1}(U, W) = UW$$

They say: use multivariate method so make up

$$W = Y_1 + Y_2 \qquad Y_2 = W - UW \Rightarrow h_2^{-1}(U, W) = W - UW$$

Tips:

given $U = X+Y$, $U = X$ or $W = Y$

given $U = \frac{X}{h(X,Y)}$, choose $W = h(X, Y)$

given $U = XY$, choose $W = Y$ or $V = X$

$$J = \det \begin{array}{c|cc} & y_1 & y_2 \\ u & w & -w \\ w & u & 1-u \end{array} = w(1-u) + uw = w$$

$|J| = |w| = ?$ is w + or $-$?

$$y_1, y_2 > 0$$

$$\left.\begin{array}{l} y_1 = uw \Rightarrow \begin{cases} u > 0, & w > 0 \\ u < 0, & w < 0 \end{cases} \\ y_2 = w(1-u) \Rightarrow \begin{cases} w > 0, & u < 1 \\ w < 0, & u > 1 \end{cases} \end{array}\right\} \Rightarrow w > 0, \quad u < 1$$

Finish!

Sample

Random sample

- $Y_1, Y_2, Y_3, Y_4, \ldots Y_n$
- a specific set of random vars

Sample observations

- 1, 2, 3...
- outcomes

Stats:

- $\bar{Y}$ = mean
- S^2 = variance = $\frac{1}{n-1} \sum_i (Y_i - \bar{Y})^2$

$$\bar{Y} = \frac{\sum Y_i}{n}$$

Proof: $E[\bar{Y}] = E[Y]$

$$LHS = E[\frac{1}{n}\sum_i Y_i] = \frac{1}{n}\sum_i E[Y_i] = \frac{1}{n}\sum_i E[Y]$$

$$= \frac{1}{n} \times n \; E[Y] = E[Y] \quad \square$$

Proof: $Var(\bar{Y}) = \frac{1}{n} Var(Y)$

$$LHS: \; Var(\frac{1}{n}\sum_i E[Y_i]) = \frac{1}{n^2} Var(\sum_i E[Y_i])$$

$$= \frac{1}{n^2} \times \left(\sum_i Var(Y_i) - 2\sum_{i,j<i} Cov(Y_i, Y_j)\right) \qquad (iid)$$

$$= \frac{1}{n^2} \times \left(\sum_i Var(Y_i) + 0\right)$$

$$= \frac{1}{n^2} \times n \times Var(Y) = \frac{1}{n} Var(Y) \quad \square$$

If $Y_1, Y_2, Y_3, \dots Y_n \sim \text{normal}(\mu, \sigma^2)$

$\bar{Y} \sim \text{normal}(\mu, \frac{\sigma^2}{n})$

- $E[\bar{Y}-\mu] = 0$
- $Var(\bar{Y}-\mu) = Var(\bar{Y}) + Var(-\mu) - 2 * Cov(\bar{Y}, \mu) = \frac{\sigma^2}{n} + 0 + 0$

- $E\left[\frac{\bar{Y}-\mu}{\frac{\sigma}{\sqrt{n}}}\right] = \frac{1}{\frac{\sigma}{\sqrt{n}}} E[\bar{Y}-\mu] = \frac{\sqrt{n}}{\sigma} * 0 = 0$

- $Var\left[\frac{\bar{Y}-\mu}{\frac{\sigma}{\sqrt{n}}}\right] = \frac{1}{\frac{\sigma^2}{n}} Var[\bar{Y}-\mu] = \frac{n}{\sigma^2} * \frac{\sigma^2}{n} = 1$

Law of large number

if $n > 30$, consider $\sigma = s$

$\bar{Y} = \frac{\sum_{i=1}^{n} Y_i}{n}$

pop. mean

$\lim_{n \to \infty} P(|\bar{Y} - \mu| < \varepsilon) = 1$ for any $\varepsilon > 0$

sample mean

t - distribution (Z dist with unknown variance)

$Z \sim \text{Normal}(0,1)$

$W \sim \chi^2_v$

$t_v \sim \frac{Z}{\sqrt{\frac{W}{n+1}}}$

unknown

$Y \sim n(\mu, \sigma^2)$

s, n given

$P(Y < k)$

$= P\left(\frac{Y-\mu}{\frac{s}{\sqrt{n}}} < \frac{k-\mu}{\frac{s}{\sqrt{n}}}\right)$

$= P\left(t_{n-1} < \frac{k-\mu}{\frac{s}{\sqrt{n}}}\right)$

$\chi^2_v \sim \frac{(n-1)s^2}{\sigma^2}$ where s is the variance of a normal dist

$Z \sim n(0,1)$

$Q = Z^2 \sim \chi^2_1$

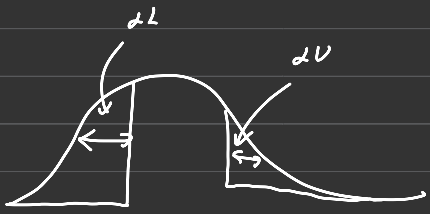
F-distribution

$W_1 \sim \chi^2_{v_1}$ $\quad$ $W_2 \sim \chi^2_{v_2}$

$$F_{(v_1, v_2)} \sim \frac{\frac{W_1}{v_1}}{\frac{W_2}{v_2}}$$

$$\frac{s_A^2}{s_B^2} \sim F_{(n_A - 1, n_B - 1)}$$

Tut Q2, Q3 ignore
Solve 1, 4, 5, 6, 7



$$P(Y < F^{\alpha L}_{(c,d)}) = \alpha$$

$$P(Y > F^{\alpha U}_{(c,d)}) = 1 - \alpha$$

$$F^{\alpha U}_{(c,d)} = \frac{1}{F^{\alpha L}_{(c,d)}}$$

Central limit theorem

$$\lim_{n\to\infty} W = \lim_{n\to\infty} \sum_{i=1}^{n} Y_i \sim Normal(n\mu_Y, n\sigma_Y^2) \qquad n > 30$$

$$\lim_{n\to\infty} \bar{Y} = \lim_{n\to\infty} \frac{1}{n}\sum Y_i \sim Normal\left(\mu_Y, \frac{\sigma_Y^2}{n}\right)$$

Doesn't apply to

$Y = \sum X_i \sim bi(n, p)$

$$Var(X_i) = 1^2 * p - E[X_i]^2$$
$$= p - p^2 = p(1-p)$$

$E[Y] = np$

$E[\bar{X}] = E[\frac{1}{n}\sum X_i]$

$= \frac{1}{n}\sum E[X_i] = \frac{1}{n} np = p$

$Var(\bar{X}) = Var(\frac{1}{n}\sum X_i)$

$= \frac{1}{n^2} Var(\sum_i X_i)$

$= \frac{1}{n^2}\left(\sum Var X_i + 2\sum_{1 = i < j}^{n} Cov(X_i, X_j)\right)$

$= \frac{1}{n^2}(\sum_i Var X_i) = \frac{1}{n^2} * n * p(1-p) = \frac{p(1-p)}{n}$

Binomial aproximation

$$X \sim \text{binomial}(n, p) \approx n(np, np(1-p)) = X_*$$

$$\text{if} \quad E[X] \geq 5 \;, \quad n(1-p) \geq 5 \;, \quad 0 < p - 3\sqrt{\frac{p(1-p)}{n}} \quad \text{and} \quad \pi + 3\sqrt{\frac{p(1-p)}{n}} < 1$$

$$P(X \leq x) \approx P(X_* \leq x + \tfrac{1}{n})$$
$$P(X \geq x) \approx P(X_* \geq x - \tfrac{1}{n})$$

Estimators

bias $= B(\hat{\theta}) = E[\hat{\theta}] - \theta$

- want to min. bias

$$\begin{aligned}
MSE(\hat{\theta}) &= E[(\hat{\theta} - \theta)^2] \\
&= E[\hat{\theta}^2 - 2\hat{\theta}\theta + \theta^2] \\
&= E[\hat{\theta}^2] - 2\theta E[\hat{\theta}] + \theta^2 \\
&= E[\hat{\theta}^2] - E[\hat{\theta}]^2 + E[\hat{\theta}]^2 - 2\theta E[\hat{\theta}] + \theta^2 \\
&= Var(\hat{\theta}) + B(\hat{\theta})^2
\end{aligned}$$